C. Surgery

Closely related to handlebodies is surgery

let $f: M \to \mathbb{R}$ be a Morse function

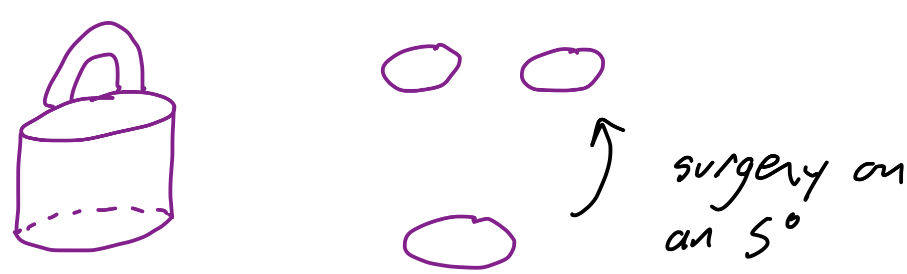
and $[a,b]$ an iterval with one critical point of index k in $f^{-1}((a,b))$

then $f^{-1}([a,b]) \cong (f^{-1}(a) \times [a, a+\varepsilon]) \cup$ k-handle attached to $f^{-1}(a) \times \{a+\varepsilon\}$

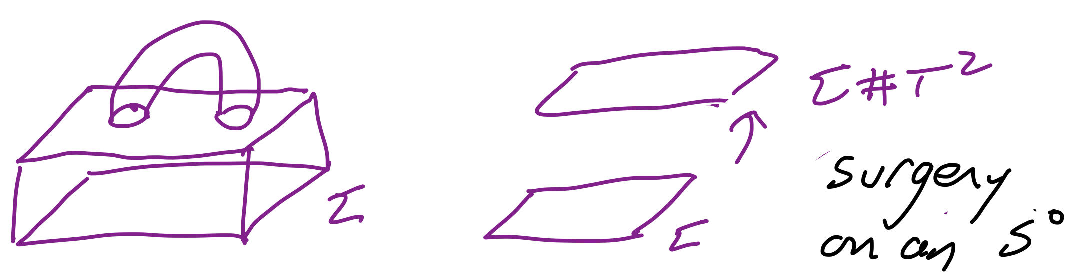
we say $\partial_+ f^{-1}([a,b]) = f^{-1}(a)$ is obtained from $\partial_- f^{-1}([a,b]) = f^{-1}(b)$ by surgery on a $S^{k-1} \subset f^{-1}(a)$ with framing given by the attaching data for the k-handle

example:

2D 1-handle



3D 1-handle



3D 2-handle

another definition of surgery

if S^k is an embedded k-sphere in M^n

and S^k has a neighborhood N diffeo to $S^k \times D^{n-k}$

choose a diffeo $\phi: S^k \times D^{n-k} \to N$ (i.e. framing)

let $M' = \overline{M^n - \mathrm{im}\,\phi} \cup D^{k+1} \times S^{n-k-1} /\sim$

where $\partial(D^{k+1} \times S^{n-k-1}) = S^k \times S^{n-k-1}$

is glued to $\partial(\overline{M^n - \mathrm{im}\,\phi})$ by $\phi|_\partial$

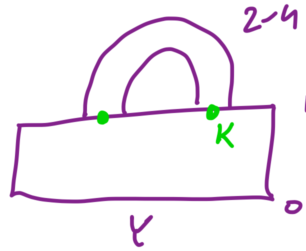
we say M' is obtained from M by ϕ-framed surgery on $S^k \subset M$

we call this a k-surgery

note: from above the level sets of a Morse function change by a $(k-1)$-surgery as the function passes an index k critical point let's consider the situation of surgery on a knot K in a 3-manifold Y

if Y' is obtained from Y by surgery on K

the we see



2-handle attached by $\phi: \partial_- h^2 \to Y \times \{1\}$
$\overset{\shortparallel}{S^1 \times D^2}$

so $Y' = \overline{Y - \text{image}(\phi)} \cup_\phi D^2 \times S^1$

ϕ sends $\partial D^2 \times \{pt\}$ to $\phi(S^1 \times \{pt\})$

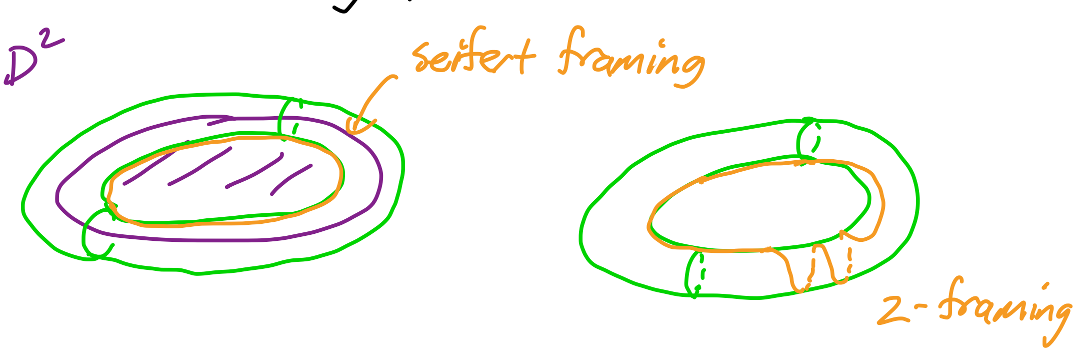
i.e. meridian of the new solid torus is glued to the longitude of K determined by the framing $\phi(S^1 \times \{pt\})$

if K is null-homologous, then $K = \partial\Sigma$ some embedded surface exercise: show this!

the surface gives K a framing call this the Seifert framing exercisé: show this is well-defined

now the $n \in \mathbb{Z}$ framing on K will be the Seifert framing plus n meridians

eg $K = \partial D^2$



so surgery on K in Y is determined by an integer

Theorem 3 (Lickorish, Wallace ~1960):

> every closed, oriented 3-manifold can be obtained by surgery on a link in S^3

There are 2 ways to prove this:

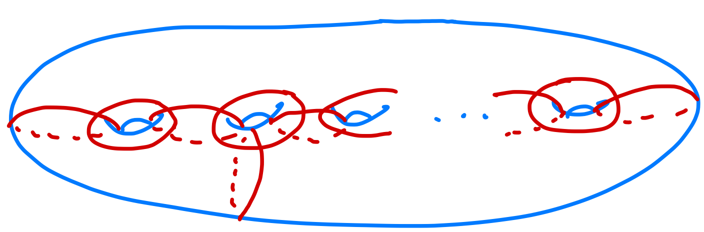
1) (Lickorish) uses Heegaard splittings and the fact that the diffeomorphisms of a surface are isotopic to compositions of Dehn twists.

Here is a sketch:

Fact (Dehn-Lickorish):

> any orientation preserving homeomorphis of a surface is isotopic to a composition of Dehn twists

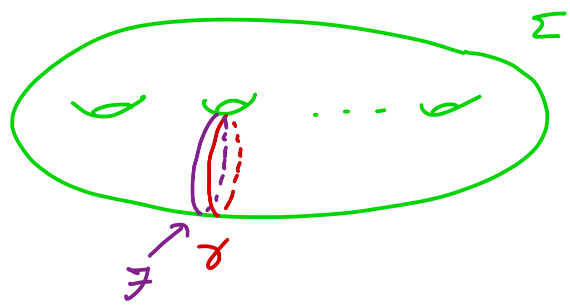
in fact, Humphries showed you only need Dehn twists about



The second result we need is

lemma:

let $\Sigma \subset M^3$ be a surface and M' be the result of cutting M^3 along Σ and regluing by a Dehn twist $\tau_\gamma^{\pm 1}$ then $M' \cong$ the result of $\mathcal{F} \mp 1$ surgery on γ, where is the framing on γ induced by Σ



here is one more simple lemma

lemma:

Suppose the homeomorphism $\phi: \Sigma \to \Sigma$ is a composition $\phi = \phi_1 \circ \phi_2$ of two homeomorphisms

let $\Sigma \subset M$ and $N = \Sigma \times [-1, 1]$ be a nbhd of Σ w/ $\Sigma = \Sigma \times \{0\}$

set $\Sigma' = \Sigma \times \{1/2\}$

let $M' = M$ cut along Σ and reglued by ϕ and $M'' = M$ cut along Σ and Σ' and reglued by ϕ_2 along Σ' and ϕ_1 along Σ

Then $M' \cong M''$

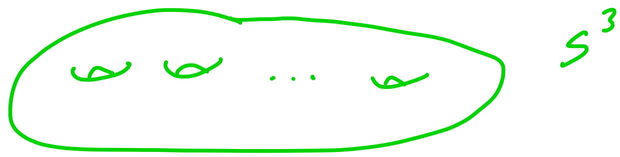
You can find the proof of these 2 lemmas in my notes on 3-manifold topology

Proof of Th$^{\underline{m}}$ 3:

given M then $\exists$ a genus g Heegaard splitting

$$M = V_1 \cup_\Sigma V_2$$

by stabilizing we know S^3 has a genus g splitting



$$S^3 = V_1 \cup_\Sigma V_2$$

so $\exists$ some orien. pres. homeomorphism $\phi : \Sigma \to \Sigma$ s.t.

$M = S^3$ cut along Σ and reglued by ϕ

now Dehn-Lickorish $\Rightarrow$ $\phi = \tau_{\gamma_1}^{\varepsilon_1} \circ \dots \circ \tau_{\gamma_n}^{\varepsilon_n}$

for some curves γ_i and $\varepsilon_i = \pm 1$

let $N = \Sigma \times [-1,1]$ be a nbhd of $\Sigma \subset S^3$

now let $\Sigma_i = \Sigma \times \{\frac{1}{i}\}$ $\quad i = 1, \dots n$

and think of γ_i as sitting on Σ_{n-i+1}

now $M = S^3$ cut along the Σ_i and reglued along Σ_i by $\tau_{\gamma_{n-i+1}}$ by the lemma

from above each regluing by $\tau_{\gamma_{n-i+1}}$ is a surgery on γ_{n-i+1}

$\therefore$ $M = S^3$ after surgery along $\gamma_1 \cup \dots \cup \gamma_n$ $\boxplus$

Corollary 4:

Every oriented 3-manifold bounds an oriented 4-manifold (that is made with a 0-handle and 2-handles)!

so $\Omega_3 = \{0\}$

Proof: since $S^3 = \partial B^4$ and you can do surgeries by attaching 2-handles we see that every oriented 3-manifold bounds an oriented 4-manifold

2) Second way to prove Thm 3. Wallace uses the fact that $\Omega_3 = \{0\}$

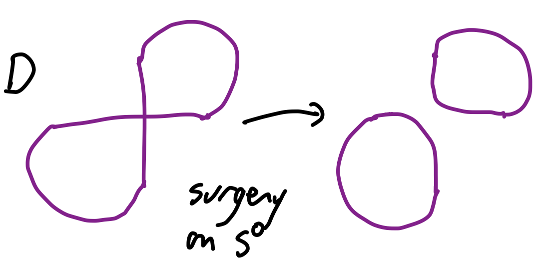
Facts: 1) M^3 orientable $\Rightarrow TM \cong M \times \mathbb{R}^3$

2) by the jet transversality result discussed after we stated the Whitney trick we know M immerses in $\mathbb{R}^5$

the double points will be a union of S^1's

4) You can change M by surgery to get a manifold M' embedded in $\mathbb{R}^5$

idea: in 2D



<u>note:</u> there is a cobordism W^5

$$\partial W = -M \cup M'$$

5) <u>exercise:</u> if $Y^n \subset X^{n+2}$ is null-homologous then there is an embedded $\Sigma^{n+1} \subset X$ such that $Y = \partial \Sigma^{n+1}$

right way to find Seifert surfaces!

<u>Hint:</u> the long exact sequence of (X,Y) gives

$$H_{n+1}(X,Y) \xrightarrow{\partial} H_n(Y) \to H_n(X)$$

$[Y]$ generates $H_n(Y) \cong \mathbb{Z}$ but is 0 in $H_n(X)$

so $\exists\, c \in H_{n+1}(X,Y)$ s.t. $\partial c = [Y]$

can think of

$$c \in H_{n+1}(X, \underbrace{N}_{\text{nbhd of } Y}) \cong H_{n+1}(\underbrace{\overline{X - N}}_{X_Y}, \partial X_Y)$$

Poincaré Dual of c

$$P.D.(c) \in H^1(X_Y)$$

Brown representation th^m says

$$H^1(X_Y) \cong \underbrace{[X_Y, S^1]}_{\text{homotopy classes of maps } X_Y \to S^1}$$

so $P.D.(c) = [f]$ some $f : X_Y \to S^1$

can make f and $f|_{\partial X_Y} \pitchfork pt$, then $\Sigma = f^{-1}(pt)$ is an $(n+1)$-manifold in X_Y

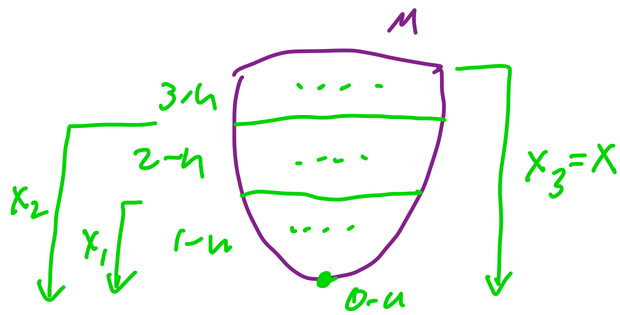
show $[\Sigma] = c$ and Σ can be extended so that $\partial \Sigma = Y$

6) so $\exists\, W' \subset S^5$ s.t. $W' = \partial M'$ and $X = W \cup W'$ is a 4-manifold

s.t. $\partial X = M$ so $\Omega_3 = \{0\}$

7) <u>Claim</u>: we can assume X has only a 0-handle and some 2-handles given this M is obtained from $S^3 = \partial(0\text{-}h)$ by surgery on some link from our discussion above!

for the claim note



now $X_1 = (0\text{-}h) \cup (1\text{-handles})$

<u>exercise</u>: 1) $(0\text{-}h) \cup$ one 1-handle $= S^1 \times D^3$

2) $X_1 = \natural_k S^1 \times D^3$

boundary sum

so $\partial X_1 = \#_k S^1 \times S^2$

3) if U an unknot in ∂B^4 adding 0-framed 2-handle gives $S^2 \times D^2$

so if U_k is the k-component unlink we can attach 2-handles to get

$X' = \natural_k S^2 \times D^2$

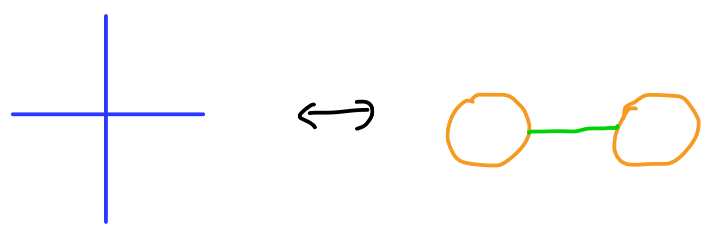
with $\partial X' = \#_k S^1 \times S^2$

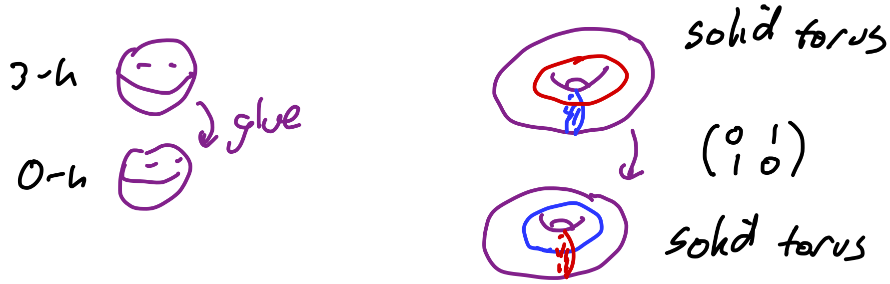
$\therefore$ replace X_1 with X' to get a new 4-manifold with no 1-handles and $\partial = M$, still call it X

now get rid of 3-handles by turning the handlebody upside down and arguing in the same way ✓

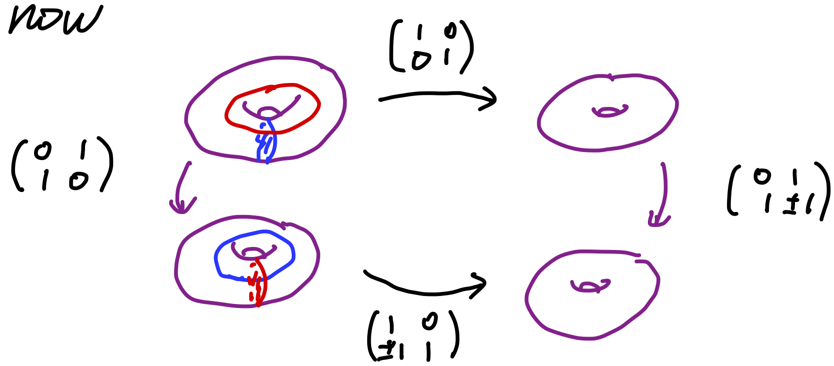
example:

1) a Heegaard diagram for S^3 is





now



note the horizontal maps extend over tori and so give a diffeomorphis

i.e. $S^3 \cong 2^{nd}$ picture

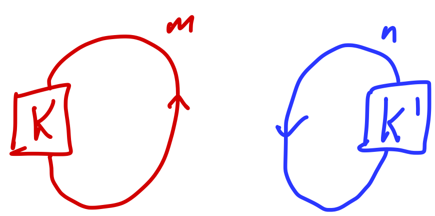
but bottom solid torus can be thought of as the complement of the unknot in S^3

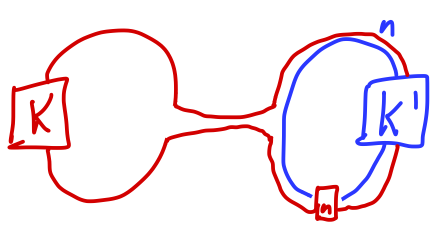
so $S^3 = (S^3 - \text{nbhd unknot}) \cup$ (solid torus glued in with meridian going to $(1,\pm 1)$-curve)

that is $S^3 \cong$ 

2) Show the S^1-bundle over S^2 with Euler number n is

note: since surgery on knots corresponds to the change in the boundary when adding 4D 2-handles we can "handle slide"





exercise: Show the new framing on K is $n + m + 2lk(K, K')$

Hint: draw the framing curve for K and push it over K' too

(this exercise will be easy once we discuss 4-manifolds more)

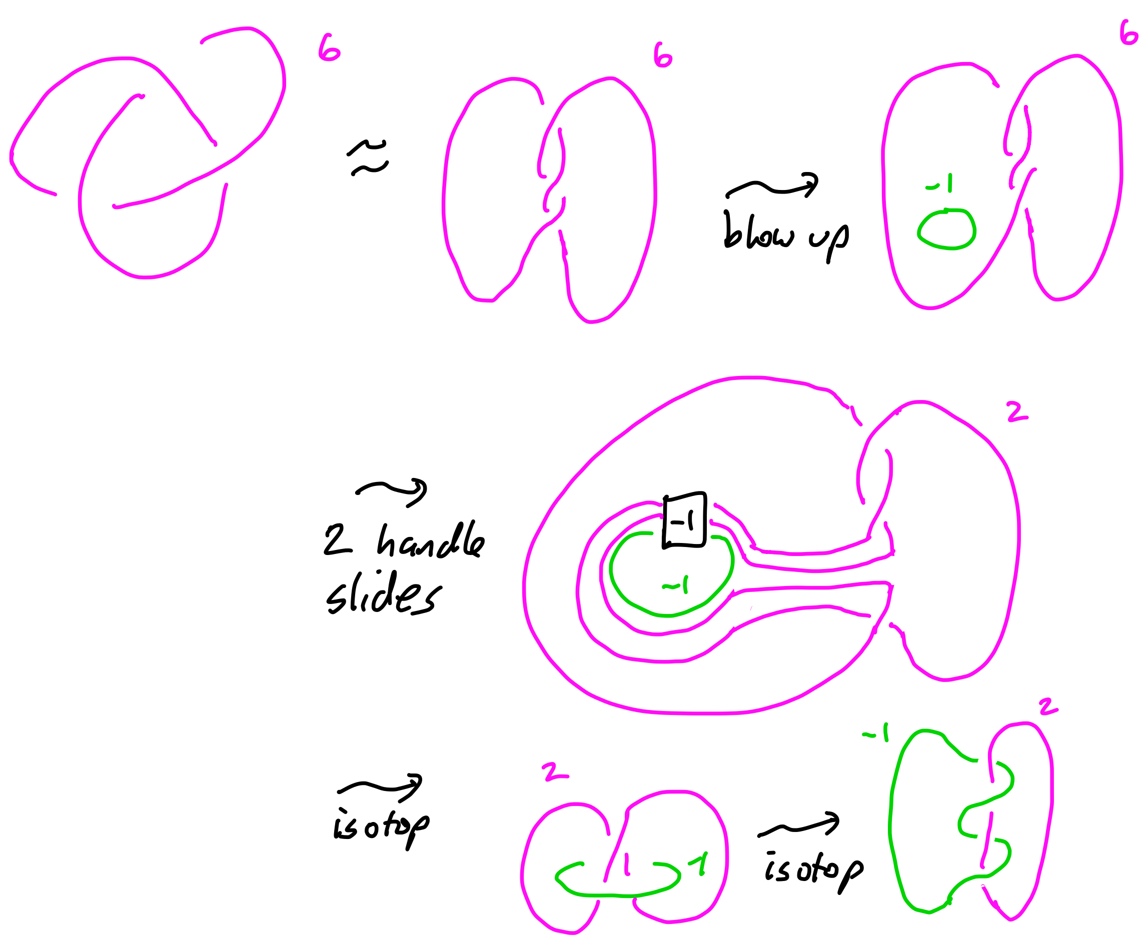
Theorem 5 (Kirby calculus):

> Two surgery descriptions of a 3-manifold represent diffeomorphic 3-manifolds iff they are related by blow ups and handle slides

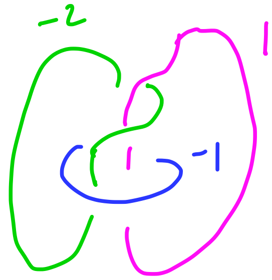
a blowup is adding a disjoint $\bigcirc^{-1}$ to the diagram

this theorem requires Cerf theory, so we will prove it later

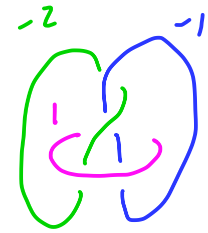
example:



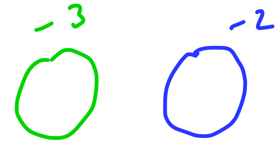
$\xrightarrow{\text{blow-up + Isotop}}$



$\xrightarrow{\text{isotop}}$



$\xrightarrow{\text{blow-down}}$



$\cong L(3,1) \# L(2,1)$

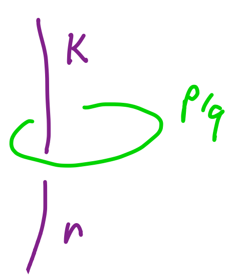
we note that "surgery" can be generalized to "Dehn surgery"

$p/q \in \mathbb{Q}$ Dehn surgery on a knot $K \subset Y$ (assume null-homologous so there is a Seifert framing)

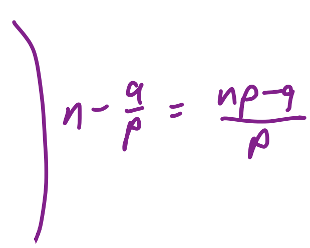
is the result of removing a neighborhood of K and gluing a solid torus back so that the meridian maps to a p/q curve (that is p meridians and q Seifert longitudes)

if p/q not an integer this does not correspond to handle attachment, as "normal" surgeries do

Fact (Slam dunk):



$\longleftrightarrow$

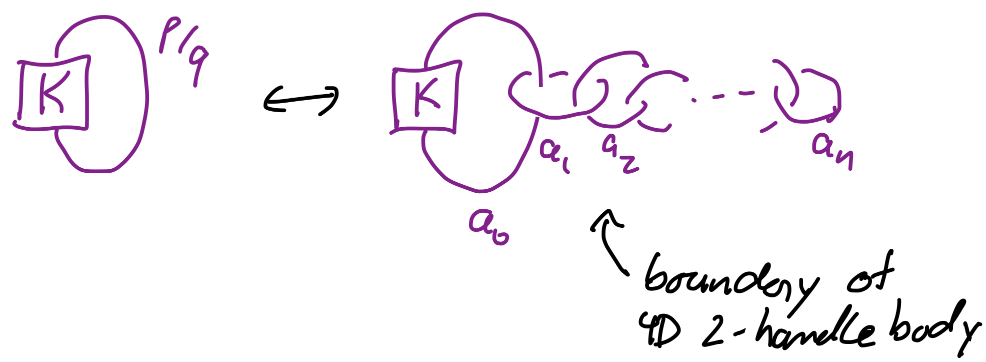


$n \in \mathbb{Z}$

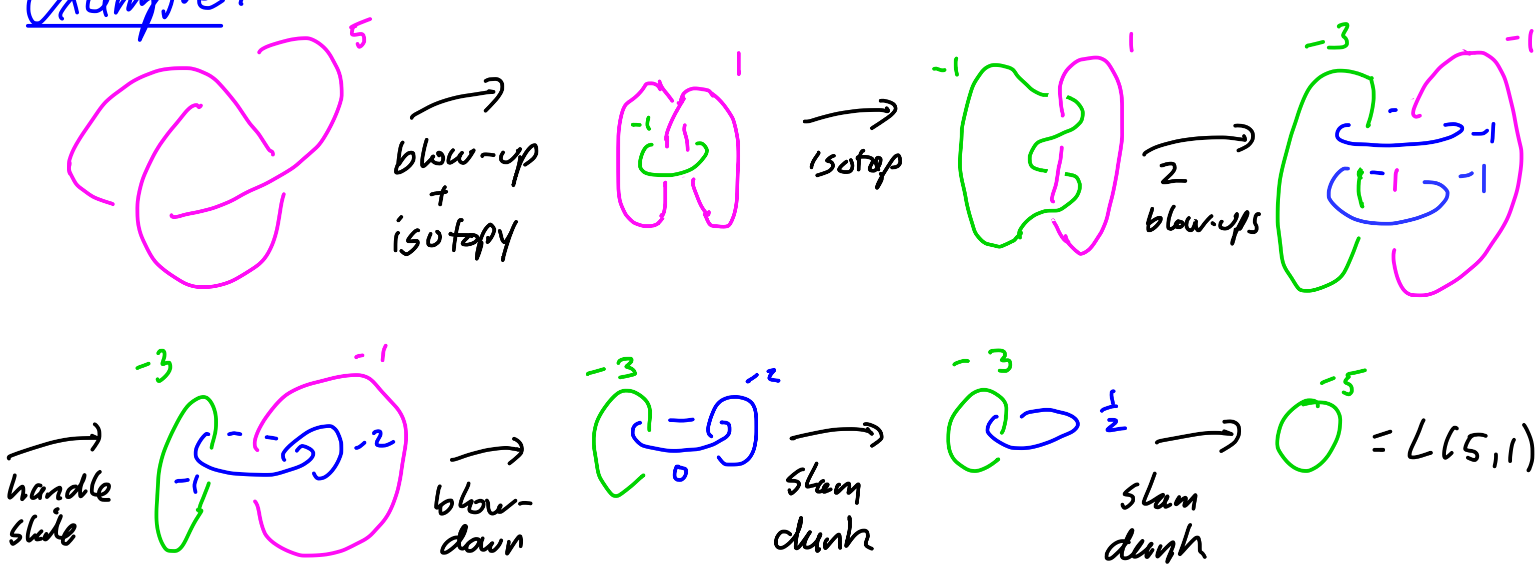
exercise: if $p/q = a_0 - \cfrac{1}{a_1 - \cdots - \cfrac{1}{a_n}}$ for $a_n \in \mathbb{Z}$, $a_i \leq -2$, $i > 0$, $a_0 = \lceil p/q \rceil$

denote by $[a_0, \dots, a_n]$

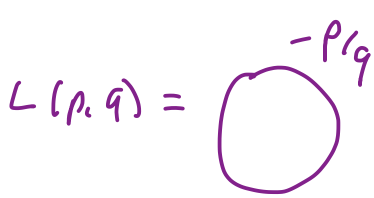
then



example:



exercise:



You can find much more about surgery and Dehn surgery in my notes on 3-manifold (on RTG website)